Formulas For Triangles

#27 of Gottschalk's Gestalts

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□ laws governing side / angle relationships of triangles

Δ consider any triangle
with vertices / angles A, B, C,
with opposite sides a, b, c,
with circumradius R;
then the following laws hold:

Δ law of sines

- $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Δ law of cosines

- $a^2 = b^2 + c^2 - 2\,b\,c\cos A$

- $b^2 = c^2 + a^2 - 2\,c\,a\cos B$

- $c^2 = a^2 + b^2 - 2\,a\,b\cos C$

Δ variations on the law of cosines

- a^2

$= b^2 + c^2 - 2\,b\,c\cos A$

$= (b+c)^2 - 4\,b\,c\cos^2\dfrac{A}{2}$

$= (b-c)^2 + 4\,b\,c\sin^2\dfrac{A}{2}$

& cyclicly

Δ law of cosines - projection

- $a = b\cos C + c\cos B$

- $b = c\cos A + a\cos C$

- $c = a\cos B + b\cos A$

Δ variations on the law of cosines - projection

- $\tan A = \dfrac{a\sin B}{c - a\cos B} = \dfrac{a\sin C}{b - a\cos C}$

& cyclicly

Δ law of tangents

- $$\frac{a-b}{a+b}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}$$
- $$\frac{b-c}{b+c}=\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}$$
- $$\frac{c-a}{c+a}=\frac{\tan\frac{1}{2}(C-A)}{\tan\frac{1}{2}(C+A)}$$

Δ law of cotangents

- $(a+b)\cot\frac{1}{2}(A+B)=(a-b)\cot\frac{1}{2}(A-B)$
- $(b+c)\cot\frac{1}{2}(B+C)=(b-c)\cot\frac{1}{2}(B-C)$
- $(c+a)\cot\frac{1}{2}(C+A)=(c-a)\cot\frac{1}{2}(C-A)$

Δ law of secants

- $a \sec \frac{1}{2}(B+C) = (b+c) \sec \frac{1}{2}(B-C)$
- $b \sec \frac{1}{2}(C+A) = (c+a) \sec \frac{1}{2}(C-A)$
- $c \sec \frac{1}{2}(A+B) = (a+b) \sec \frac{1}{2}(A-B)$

Δ law of cosecants

- $a \csc \frac{1}{2}(B+C) = (b-c) \csc \frac{1}{2}(B-C)$
- $b \csc \frac{1}{2}(C+A) = (c-a) \csc \frac{1}{2}(C-A)$
- $c \csc \frac{1}{2}(A+B) = (a-b) \csc \frac{1}{2}(A-B)$

Δ Mollweide' s equations
which are slight variations of
the law of secants
&
the law of cosecants

- $$\frac{a}{b+c} = \frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}(B-C)}$$

- $$\frac{b}{c+a} = \frac{\sin\frac{1}{2}B}{\cos\frac{1}{2}(C-A)}$$

- $$\frac{c}{a+b} = \frac{\sin\frac{1}{2}C}{\cos\frac{1}{2}(A-B)}$$

- $$\frac{a}{b-c}=\frac{\cos\frac{1}{2}A}{\sin\frac{1}{2}(B-C)}$$

- $$\frac{b}{c-a}=\frac{\cos\frac{1}{2}B}{\sin\frac{1}{2}(C-A)}$$

- $$\frac{c}{a-b}=\frac{\cos\frac{1}{2}C}{\sin\frac{1}{2}(A-B)}$$

☐ formulas for the trig fcns
of the angles of a triangle
ito the sides

Δ the 6 basic trig fcns of the 3 angles of a triangle
with vertices / angles $A,\ B,\ C,$
with opposite sides $a,\ b,\ c,$
with semiperimeter s
are given ito the sides by 18 formulas
of which the first 3 are listed below,
the remaining 15 then being easy consequences:

- $\sin A = \dfrac{2}{b\,c}\sqrt{s\,(s-a)\,(s-b)\,(s-c)}$

- $\cos A = \dfrac{b^2 + c^2 - a^2}{2\,b\,c}$

- $\tan A = 4\,\dfrac{\sqrt{s\,(s-a)\,(s-b)\,(s-c)}}{b^2 + c^2 - a^2}$

Δ also note the formulas

- $1+\cos A = \dfrac{2s(s-a)}{bc}$

& cyclicly

- $1-\cos A = \dfrac{2(s-b)(s-c)}{bc}$

& cyclicly

□ formulas for the trig fcns
of the half - angles of a triangle
ito the sides

Δ the 6 basic trig fcns of the 3 half - angles of a triangle
with vertices / angles A, B, C,
with opposite sides a, b, c,
with semiperimeter s
are given ito the sides by 18 formulas
of which the first 3 are listed below,
the remaining 15 then being easy consequences:

- $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

- $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

- $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

☐ formulas for the area of a triangle

Δ consider any triangle
with vertices / angles $A,\ B,\ C$,
with opposite sides $a,\ b,\ c$,
with altitudes $h_a,\ h_b,\ h_c$ to sides $a,\ b,\ c$,
with semiperimeter s,
with circumradius R,
with inradius r,
with exradii $r_a,\ r_b,\ r_c$ to sides $a,\ b,\ c$;
then
the area K of the triangle
is given by
the following formulas:

- $K = \frac{1}{2} a\, h_a$ & cyclicly

- $K = \frac{1}{2} b\, c \sin A$ & cyclicly

- $K = \frac{1}{2} a^2 \frac{\sin B \sin C}{\sin A}$ & cyclicly

- $K = \frac{1}{2} a^2 \frac{\sin B \sin C}{\sin (B + C)}$ & cyclicly

- $K = 2 R^2 \sin A \sin B \sin C$

- $K = 4 R s \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

- $K = 4 R r \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$

- $K = s^2 \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}$

- $K = s(s-a)\tan\frac{A}{2}$ & cyclically

- $K = \sqrt{s(s-a)(s-b)(s-c)}$ (Heron' s formula; already known to Archimedes)

- $K = \frac{abc}{4R}$

- $K = rs$

- $K = r_a(s-a)$ & cyclically

- $K = \sqrt{r\, r_a\, r_b\, r_c}$

Δ many pretty formulas
are easy consequences
of the above area formulas
eg

- $r = 4\,R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

- $s = 4\,R \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$

- $r = s \tan\frac{A}{2} \tan\frac{B}{2} \tan\frac{C}{2}$

- $\tan\frac{A}{2} = \frac{r}{s-a} = \frac{r_a}{s}$ & cyclicly

- $abc = 4Rrs$

□ some trig eqns that are true
for all angles A, B, C
such that their sum is a straight angle:
$A + B + C = 180^{o} = \pi^{r}$;
hence these formulas hold
for the angles A, B, C of any triangle

- $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

- $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

- $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

- $\cos 2A + \cos 2B + \cos 2C + 4\cos A\cos B\cos C + 1 = 0$

- $\sin 4A + \sin 4B + \sin 4C + 4\sin 2A\sin 2B\sin 2C = 0$

- $\cos 4A + \cos 4B + \cos 4C + 1 = 4\cos 2A\cos 2B\cos 2C$

- $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

- $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

- $(\cot A + \cot B + \cot C)^2 = \cot^2 A + \cot^2 B + \cot^2 C + 2$

- $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$

- $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$

- $\left(\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2}\right)^2 = \tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} + 2$

- $\sin^2 A + \sin^2 B + \sin^2 C \; = \; 2 + 2\cos A \cos B \cos C$

- $\sin^2 A + \sin^2 B - \sin^2 C \; = \; 2\sin A \sin B \cos C$
- $\sin^2 A - \sin^2 B + \sin^2 C \; = \; 2\sin A \cos B \sin C$
- $-\sin^2 A + \sin^2 B + \sin^2 C \; = \; 2\cos A \sin B \sin C$

- $\cos^2 A + \cos^2 B + \cos^2 C \; = \; 1 - 2\cos A \cos B \cos C$

- $\cos^2 A + \cos^2 B - \cos^2 C \; = \; 1 - 2\sin A \sin B \cos C$
- $\cos^2 A - \cos^2 B + \cos^2 C \; = \; 1 - 2\sin A \cos B \sin C$
- $-\cos^2 A + \cos^2 B + \cos^2 C \; = \; 1 - 2\cos A \sin B \sin C$

Metatheorem.

- the formula $F(A,\ B,\ C)$ is valid

$\Rightarrow$

- the formula $F\left(A-\pi,\ B+\frac{\pi}{2},\ C+\frac{\pi}{2}\right)$ is valid

& cyclicly

- the formula $F(-A,\ \pi-B,\ \pi-C)$ is valid

& cyclicly

- the formula $F(\pi-2A,\ \pi-2B,\ \pi-2C)$ is valid

- the formula $F(3A,\ 3B-\pi,\ 3C-\pi)$ is valid

& cyclicly

- the formula $F\left(\frac{\pi}{2}-\frac{A}{2},\ \frac{\pi}{2}-\frac{B}{2},\ \frac{\pi}{2}-\frac{C}{2}\right)$ is valid

- etc

T. if

$A,\ B,\ C$ are the angles of a triangle

then

$\sin 4A + \sin 4B + \sin 4C \ = \ 0$

$\Leftrightarrow$

$ABC \ \in \ \text{rt}\,\Delta$

∆ bioline

Archimedes of Syracuse
ca 287-212 BCE
Greek
mathematician, physicist, inventor;
considered to be one
of the three greatest mathematicians of all time,
the other two being Newton and Gauss

∆ bioline

Heron of Alexandria
fl 62 CE
Greek
mathematician, physicist, engineer

∆ bioline

Karl Brandon Mollweide
1774-1825
German
mathematician, astronomer, cartographer