An Ingenious Instantaneous Way To Invent Interesting Functions Is To Integrate & Invert

#5 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms of the Organization & Exposition of Mathematics
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Infinite Vistas Press
PVD RI
2000

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□ to define new functions by integrating & inverting:
the pattern

• $f(x) \to F(x)$ & $G(x)$

(the arrow $\to$ may be read
'produces by integration and inversion'
or simply 'produces';
only real functions considered)

(1) integrate $f(x)$

to define $y = F(x) = \int_a^x f(t)dt$

(2) then invert $y = F(x)$

to obtain $y = G(x)$

where $x, y, t \in$ real vars

note: this process will determine any
continuously differentiable function $G(x)$ with nonzero derivative;
to obtain $f(x)$, just differentiate the inverse of $G(x)$

□ a note on inverting a function

• to invert the function $y = f(x)$,
think of solving for x ito y,
and then interchange x and y
to make x the independent variable
and y the dependent variable;
in symbols

$y = f(x) \Leftrightarrow x = f^{-1}(y)$

$x = f(y) \Leftrightarrow y = f^{-1}(x)$

thus writing

$y = f^{-1}(x)$ read ' wye equals ef inverse of ex'

as the inverse function of

$y = f(x)$ read ' wye equals ef of ex'

□ the basic transcendental functions of calculus
are obtainable
by integrating very simple algebraic functions
& then inverting;
various examples are given below

□ E1. $\frac{1}{x} \to \log x$ & $\exp x = e^x$

(1) integrate $\frac{1}{x}$

to define $y = \log x = \int_1^x \frac{1}{t} dt$

(2) then invert $y = \log x$

to obtain $y = \exp x = e^x$

□ E2. $\dfrac{1}{\sqrt{1-x^2}} \to \sin^{-1} x \ \ \& \ \ \sin x$

(1) integrate $\dfrac{1}{\sqrt{1-x^2}}$

to define $y = \sin^{-1} x = \int_0^x \dfrac{1}{\sqrt{1-t^2}}\,dt$

(2) then invert $y = \sin^{-1} x$

to obtain $y = \sin x$

☐ E3. $\dfrac{1}{1+x^2} \to \tan^{-1} x \ \ \& \ \ \tan x$

(1) integrate $\dfrac{1}{1+x^2}$

to define $y = \tan^{-1} x = \int_0^x \dfrac{1}{1+x^2}\,dx$

(2) then invert $y = \tan^{-1} x$

to obtain $y = \tan x$

☐ E4. $\dfrac{1}{\sqrt{x^2+1}} \rightarrow \sinh^{-1} x \;\;\&\;\; \sinh x$

(1) integrate $\dfrac{1}{\sqrt{x^2+1}}$

to define $y = \sinh^{-1} x = \int_0^x \dfrac{1}{\sqrt{t^2+1}}\,dt$

(2) then invert $y = \sinh^{-1} x$

to obtain $y = \sinh x$

☐ E5. $\dfrac{1}{1-x^2} \to \tanh^{-1} x \;\;\&\;\; \tanh x$

(1) integrate $\dfrac{1}{1-x^2}$

to define $y = \tanh^{-1} x = \displaystyle\int_0^x \frac{1}{1-t^2}\,dt$

(2) then invert $y = \tanh^{-1} x$

to obtain $y = \tanh x$

☐ a list of transcendental functions
defined by indefinite integrals
with simple algebraic integrands

• log fcn

$$\log x = \int_1^x \frac{1}{t}\,dt$$

• inv trig fcns

$$\sin^{-1}x = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt$$

$$\cos^{-1}x = \int_x^1 \frac{1}{\sqrt{1-t^2}}\,dt$$

$$\tan^{-1}x = \int_0^x \frac{1}{1+t^2}\,dt$$

$$\cot^{-1}x = \int_x^\infty \frac{1}{1+t^2}\,dt$$

$$\sec^{-1}x = \int_1^x \frac{1}{t\sqrt{t^2-1}}\,dt$$

$$\csc^{-1}x = \int_x^\infty \frac{1}{t\sqrt{t^2-1}}\,dt$$

• inv hyp fcns

$$\sinh^{-1} x = \int_0^x \frac{1}{\sqrt{t^2+1}}\,dt$$

$$\cosh^{-1} x = \int_1^x \frac{1}{\sqrt{t^2-1}}\,dt$$

$$\tanh^{-1} x = \int_0^x \frac{1}{1-t^2}\,dt$$

$$\coth^{-1} x = \int_x^\infty \frac{1}{1-t^2}\,dt$$

$$\operatorname{sech}^{-1} x = \int_x^1 \frac{1}{t\sqrt{1-t^2}}\,dt$$

$$\operatorname{csch}^{-1} x = \int_x^\infty \frac{1}{|t|\sqrt{1-t^2}}\,dt$$

• elliptic integrals of the first kind

$$\int_0^x \frac{1}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\,dt$$

• elliptic integrals of the second kind

$$\int_0^x \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,dt$$

• elliptic integrals of the third kind

$$\int_0^x \frac{1}{(1-nt^2)\sqrt{1-t^2}\sqrt{1-k^2t^2}}\,dt$$

note: elliptic functions are
the inverse functions of elliptic integrals

□ the gudermannian & its inverse
as indefinite integrals

$$\mathrm{gd}\,x = \int_0^x \mathrm{sech}\,t\,dt = \tan^{-1}\sinh x$$

$$\mathrm{gd}^{-1}x = \int_0^x \sec t\,dt = \sinh^{-1}\tan x = \ln(\sec x + \tan x)$$

☐ a list of transcendental functions defined by indefinite integrals with transcendental integrands

- the exponential integral $$\operatorname{Ei} x = \int_{-\infty}^{x} \frac{\exp t}{t}\, dt$$

- the logarithmic integral $$\operatorname{Li} x = \int_{0}^{x} \frac{1}{\log t}\, dt$$

- the sine integral $$\operatorname{Si} x = \int_{0}^{x} \frac{\sin t}{t}\, dt$$

- the cosine integral $$\operatorname{Ci} x = -\int_{x}^{\infty} \frac{\cos t}{t}\, dt$$

- the hyperbolic sine integral $\quad \operatorname{Shi} x = \int_0^x \frac{\sinh t}{t}\, dt$

- the hyperbolic cosine integral

$$\operatorname{Chi} x = \int_0^x \frac{\cosh t - 1}{t}\, dt + \log x + \gamma$$

wh γ = Euler' s constant = $\lim_{n \to \infty} (H_n - \log n) = 0.5772156649 +$

wh H_n = the nth partial sum of the harmonic series

wh $n \in$ pos int var

- the Fresnel sine integral $\quad S(x) = \int_0^x \sin \frac{\pi}{2} t^2\, dt$

- the Fresnel cosine integral $\quad C(x) = \int_0^x \cos \frac{\pi}{2} t^2\, dt$

- the error function $\quad \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$