All Good Things Come In Three's

#42 of Gottschalk's Gestalts

A Series Illustrating Innovative Forms
of the Organization & Exposition
of Mathematics
by Walter Gottschalk

Infinite Vistas Press
PVD RI
2001

500 Angell St #414
Providence RI 02906

GG42-2

□ the three classical means

• the arithmetic mean of a and b

$$= \frac{a+b}{2}$$

= the logarithm of the geometric mean of the exponentials of a and b

• the geometric mean of a and b

$$= \sqrt{ab}$$

= the exponential of the arithmetic mean of the logarithms of a and b

• the harmonic mean of a and b

$$= \frac{2ab}{a+b}$$

= the reciprocal of the arithmetic mean of the reciprocals of a and b

□ the rule of three
is
a rule for solving a simple proportion problem
that appeared
in the early history of mathematics
and continued into more recent even modern times;
the idea goes back to the ancient Egyptians;
the explicit rule popped up in China and India ca 600 CE,
and had spread itself in print
across Western Europe by ca 1500;
no doubt many merchants around the world
have been using the rule more or less automatically
for as long as commerce has existed;
the name comes from the fact that
three quantities are given
and one is to be determined;
in modern algebraic garb
the rule of three
is the following equivalence
whose LHS is
a proportion = an equality of ratios

$$x : a = b : c \iff x = (ab) / c$$

a somewhat weaker formulation of the rule of three
is that
in a proportion
the product of the means
equals
the product of the extremes

• double rule of three

a double rule of three
(there are many forms)
contains several
direct and inverse variations;
here is a generalized modernized example
whose prototype appeared in the book
Liber abaci (1202) (Latin) (= Book of Calculations)
by
Leonardo of Pisa
= Leonardo Fibonacci
= Fibonacci

Problem. x horses eat y barley quarts in z days

whence $f(x, y) = z$;

given $f(a, b) = c$, find $f(A, B)$

Solution. call $f(A, B) = C$;

now in the nature of the problem

$$f(kx, y) = \frac{1}{k} f(x, y)$$

&

$$f(x, ky) = kf(x, y);$$

hence

$$
\begin{aligned}
& C \\
= \ & f(A, B) \\
= \ & f\left(\frac{A}{a}a, \frac{B}{b}b\right) \\
= \ & \frac{a}{A}\ \frac{B}{b} f(a, b) \\
= \ & \frac{aBc}{Ab}
\end{aligned}
$$

&

$$C = \frac{aBc}{Ab}$$

which could also be written

$$\frac{a}{A} \times \frac{B}{b} = \frac{C}{c}$$

which shows the nature of the variations

• to see how the single rule of three
is related to a double rule of three,
the earlier version of the single rule of three
needs to be restated in the following form;
let us take a simple example

Problem. x apples cost y cents
whence $f(x) = y$;
given $f(a) = b$, find $f(A)$

Solution. call $f(A) = B$;
now in the nature of the problem
$f(kx) = kf(x)$;

hence

$$
\begin{aligned}
&B \\
&= f(A) \\
&= f\left(\frac{A}{a}a\right) \\
&= \frac{A}{a}f(a) \\
&= \frac{Ab}{a}
\end{aligned}
$$

&

$$B = \frac{Ab}{a}$$

which also could be written

$$\frac{A}{a} = \frac{B}{b}$$

which is the earlier recognized kind of proportion

• I first heard of the double rule of three in the following common measure sestet:

He thought he saw a Garden-Door
That opened with a key:
He looked again, and found it was
A Double Rule of Three:
'And all its mystery,' he said,
'Is clear as day to me!'

from Sylvie and Bruno
by Lewis Carroll

□ pythagorean triples

a pythagoreran triple
= df an ordered triple (a, b, c)
of positive integers st
$a^2 + b^2 = c^2$
or equivalently
an ordered triple (a, b, c)
of positive integers a, b, c that are
the lengths of the legs and the hypotenuse
of a right triangle
whence the name
pythagorean triple
which is suggested by the name
pythagorean theorem

every triple (a, b, c) st

$a = \lambda(m^2 - n^2)$

$b = 2\lambda mn$

$c = \lambda(m^2 + n^2)$

wh m, n, λ are positive integers with $m > n$

is a pythagorean triple;

conversely

every pythagorean triple (a, b, c)

is of this form

with the possible interchange of the first two entries

for a pythagorean triple (a, b, c)
not both a and b are odd
&
any common factor of two of a, b, c
is also a factor of the third
and when divided out the resulting quotients
still constitute a pythagorean triple
&
(b, a, c) is also a pythagorean triple;
this suggests the definition
a primitive pythagorean triple
= df a pythagorean triple (a, b, c) st
b is even & a and b are relatively prime;
any pythagorean triple is obtainable from
a unique primitive pythagorean triple
by multiplication thruout by a positive integer
and
by possible interchange of the first two entries

every triple (a, b, c) st

$a = m^2 - n^2$

$b = 2mn$

$c = m^2 + n^2$

wh m and n are relatively prime positive integers

of different parity and with $m > n$

is a primitive pythagorean triple;

conversely

every primitive pythagorean triple (a, b, c)

is uniquely of this form

□ the three-squares theorem
of elementary number theory

a positive integer n is
the sum of the squares of three integers
if and only if
there are no nonnegative integers r and s such that

$$n = 4^r(8s+7)$$

GG42-15

☐ three primes & Goldbach

• in a letter written to Euler in 1742
the German-Russian mathematician
Christian Goldbach (1690-1764)
conjectured that
every integer greater than 5
is the sum of three primes;
Euler rephrased the conjecture equivalently as
every even integer greater than 2
is the sum of two primes
which is the form in which
Goldbach's Conjecture is now usually stated

• the Second/Other Goldbach Conjecture
states that
every odd integer greater than 5
is the sum of three primes

• as of 2000
neither conjecture has been proved or disproved

□ the 3n+1 problem

start with any positive integer n;
if n is even, divide n by 2;
if n is odd, multiply n by 3 and add 1
which produces an even integer 3n+1;
repeat the process on the integer obtained
and continue in order to form a sequence;
prove or disprove the conjecture
that the resulting sequence
always reaches the number 1
(and then the sequence will cycle as 1, 4, 2, 1, etc);
this problem is unsolved at the present time
(May 2001);
the conjecture has been verified by actual calculation
up to astronomical numbers

☐ the three cube roots of unity

- 1

- $\omega = \frac{1}{2}\left(-1 + i\sqrt{3}\right)$

- $\omega^2 = \overline{\omega} = \frac{1}{2}\left(-1 - i\sqrt{3}\right)$

□ three little formulas
relating the three most important
numbers in mathematics
that are designated by letters:
π, e, i

$$e^{\pi i} + 1 = 0$$

$$i^{i} = e^{-\frac{\pi}{2}} \qquad \text{(pv)}$$

$$\sqrt[i]{i} = e^{\frac{\pi}{2}} \qquad \text{(pv)}$$

these three formulas are important enuf
to be put in limerick form

'Ee to the pie eye plus won
Goes poof' is a benison
For it wraps up a lot
In a very small spot
And proves math is always great fun.

Georgie Porgie said 'Hi!
The principal ith power of i
Is the number e to
Minus π over 2
But I cannot begin to tell why.'

Georgie Porgie said 'Hi!
The principal ith root of i
Is the number e to
Plus π over 2
But I cannot begin to tell why.'

□ the Q & O multiplicative triplets
aka
quaternion & octonion multiplications made easy

• the quaternion number system $\mathbb{H}$
is by definition & a little proof
a 4-dimensional real normed conjugated
noncommutative associative
linear division algebra
with bilinear multiplication
&
with three basic unit quaternions (besides unity)

$$\mathrm{i},\ \mathrm{j},\ \mathrm{k}$$

whose products satisfy the condition:
the ordered triple $(\mathrm{i},\ \mathrm{j},\ \mathrm{k})$ is a cyclic system
viz

$$\mathrm{i}^2 = -1$$

$$\mathrm{j}^2 = -1$$

$$\mathrm{k}^2 = -1$$

$$\mathrm{i}\,\mathrm{j} = \mathrm{k} \ \ \& \ \ \mathrm{j}\,\mathrm{i} = -\mathrm{k}$$

$$\mathrm{j}\,\mathrm{k} = \mathrm{i} \ \ \& \ \ \mathrm{k}\,\mathrm{j} = -\mathrm{i}$$

$$\mathrm{k}\,\mathrm{i} = \mathrm{j} \ \ \& \ \ \mathrm{i}\,\mathrm{k} = -\mathrm{j}$$

• the octonion number system $\mathbb{O}$
is by definition & a little proof
an 8-dimensional real normed conjugated
noncommutative nonassociative
linear algebra
with bilinear multiplication
&
with seven basic unit octonions (besides unity)

$$e_n \quad (n \in \underline{7})$$

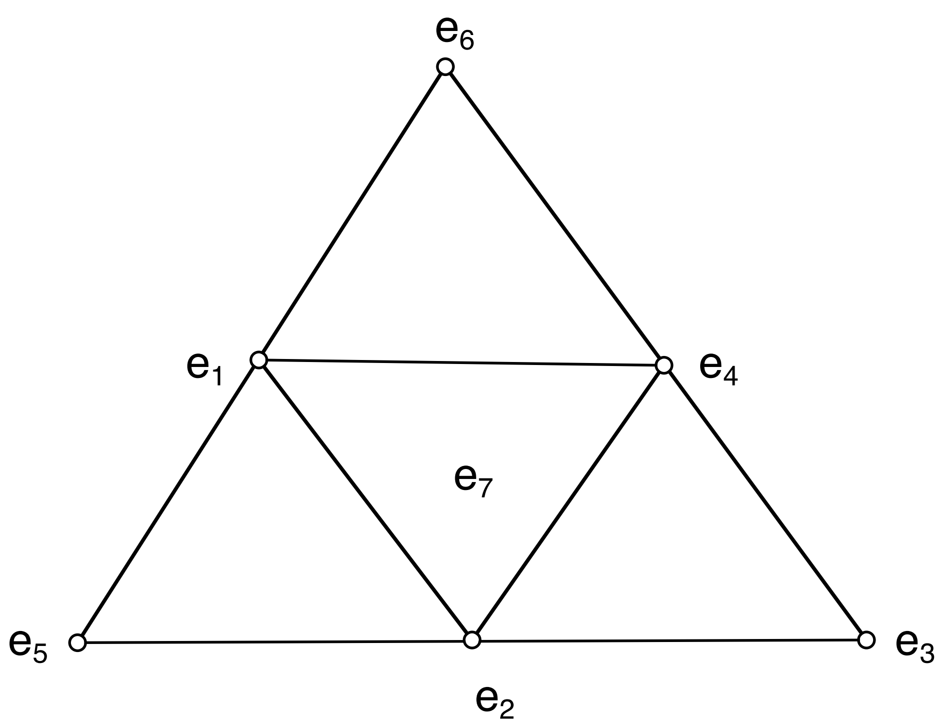
such that
each of the following seven ordered triples
is a cyclic system:

$$e_1 \quad e_2 \quad e_4$$
$$e_1 \quad e_3 \quad e_7$$
$$e_1 \quad e_5 \quad e_6$$
$$e_2 \quad e_3 \quad e_5$$
$$e_2 \quad e_6 \quad e_7$$
$$e_3 \quad e_4 \quad e_6$$
$$e_4 \quad e_5 \quad e_7$$

starting with any of the above triples
and repeatedly adding 1 to the subscripts mod 7
will yield all triples in the given cyclic order

GG42-22

a geometric mnemonic
for the above seven cyclic systems
is based on an equilateral triangle
as shown below;
the seven basic nonunity octonions
are distributed at
the three vertices,
the centroid,
the three side-midpoints
as indicated on the diagram;
there are seven 'lines'
viz
the three sides,
the three medians,
the curvilinear midpoint triangle;
think of the sides of the original triangle
and the curvilinear midpoint triangle
as oriented positively= in the counterclockwise direction;
think of the three medians as directed
from vertex to centroid to opposite side-midpoint;
each pair of units lies on just one line
and this line contains just one other unit and thus
the diagram determines a unique cyclic order
of these three units;
the seven cyclic systems
may now be readily read off the diagram



GG42-24

☐ the three sums-of-squares identities

• the sum-of-two-squares identity
= the two-squares identity
is derivable from
the multiplicative norm law
for complex numbers
viz
the norm of the product
of two complex numbers
equals
the product of the norms
of the complex numbers

representing each of two complex numbers
as the canonical basic linear combination
of two real numbers
&
substituting in
the multiplicative norm law
for complex numbers
gives
the two-squares identity
viz
for all real numbers a, b, c, d

$$\left(a^2+b^2\right)\left(c^2+d^2\right)=(ac-bd)^2+(ad+bc)^2$$

which shows inp that
the set of all sums of squares of two integers
is multiplicatively closed
= the product of two or more sums of squares of two integers
is again a sum of squares of two integers

• the sum-of-four-squares identity
= the four-squares identity
is derivable from
the multiplicative norm law
for quaternion numbers
viz
the norm of the product
of two quaternion numbers
equals
the product of the norms
of the quaternion numbers

representing each of two quaternion numbers
as the canonical basic linear combination
of four real numbers
&
substituting in
the multiplicative norm law
for quaternion numbers
gives
the four-squares identity
viz
(in compressed form)
for all complex numbers a, b, c, d

$$\left(|a|^2 + |b|^2\right)\left(|c|^2 + |d|^2\right) = \left|ac - \overline{b}d\right|^2 + \left|\overline{a}d + bc\right|^2$$

which shows inp that
the set of all sums of squares of four integers
is multiplicatively closed
= the product of two or more sums of squares of four integers
is again a sum of squares of four integers

• the sum-of-eight-squares identity
= the eight-squares identity
is derivable from
the multiplicative norm law
for octonion numbers
viz
the norm of the product
of two octonion numbers
equals
the product of the norms
of the octonion numbers

representing each of two octonion numbers
as the canonical basic linear combination
of eight real numbers
&
substituting in
the multiplicative norm law
for octonion numbers
gives
the eight-squares identity
viz
(in compressed form)
for all quaternion numbers a, b, c, d

$$\left(|a|^2 + |b|^2\right)\left(|c|^2 + |d|^2\right) = \left|ac - d\bar{b}\right|^2 + \left|\bar{a}d + cb\right|^2$$

which shows inp that
the set of all sums of squares of eight integers
is multiplicatively closed
= the product of two or more sums of squares of eight integers
is again a sum of squares of eight integers

□ terse triads

• a threefold classification of triangles:
scalene, isosceles, equilateral

• a threefold classification of triangles:
acute-angled = acute-angle = acute
right-angled = right-angle = right
obtuse-angled = obtuse-angle = obtuse

• the threefold classification of conic sections according to eccentricity e:
ellipses (e<1) [for circles e = 0]
parabolas (e=1)
hyperbolas (e>1)

• the three diagonals of a cyclic quadrilateral
(rearrange the sides in the circumscribing circle)

• the three regular tesselations of the plane by:
equilateral triangles, squares, hexagons

• the three classical construction problems of Greek geometry:
trisect an angle, duplicate a cube, square a circle by Platonic tools alone

• names of the three coordinate axes
of a rectangular coordinate system in 3-space:
x-axis, y-axis, z-axis

• names of the three coordinates of a point
wrt a rectangular coordinate system in 3-space:
x-coordinate = abscissa
y-coordinate = ordinate
z-coordinate = altitude

• the determinant three-point form
of the equation of a plane in 3-space
provided with a rectangular coordinate system
is
a fourth order determinant placed equal to 0
where
the fourth column of the determinant
consists entirely of four 1's
and
the other entries in the four consecutive rows
of the determinant
are occupied by
the coordinates of the running point
and
the coordinates of the three given points

• the three direction
angles, cosines, numbers
of a line (possibly directed) in 3-space
provided with a rectangular coordinate system

• the three-term direction ratio
of a line (possibly directed) in 3-space
provided with a rectangular coordinate system

• the three projection planes
of a line in 3-space
provided with a rectangular coordinate system

• the three axes of an ellipsoid:
the major axis, the mean axis, the minor axis

• the three kinds of reflective symmetry in 3-space:
central, axial, planar

• the scalar product of three 3-vectors
= the determinant of three 3-vectors;
the left/right vector product of three 3-vectors

• the three crossings
of a left-handed/right-handed trefoil knot
which is the simplest of all knots

• the number system to the base 3
= the base 3 number system
= the ternary number system
= the 3-ary number system
= the triadic number system
= the 3-adic number system

• the three ef number sequences:
the Fermat numbers
the Fibonacci numbers
the figurate numbers

• the threefold
sum-of-the-proper-divisors classification
of positive integers as:
abundant, perfect, deficient

• Gauss proved
when he was only nineteen year old that
every positive integer
is the sum of at most three triangular numbers

• ¿are there infinitely many prime triplets?
where a prime triplet is such as (3, 5, 7);
answer unknown at present (2000);
it is not known at present (2000)
whether there are infinitely many twin primes
where a twin prime pair is such as (3, 5)

• the three signs of real numbers:
positive, zero, negative

• the threefold IFI
exclusive & exhaustive
classification of real numbers:
integers
fractions = noninteger rational numbers
irrationals = irrational numbers

• the square root of 3
$\sqrt{3} = 1.732+$
may be called
the George Washington number
because
George Washington was born in 1732

• the cubic equation of Wallis

$$x^3 - 2x - 5 = 0$$

has the unique real root

$$x = 2.094551\cdots$$

• the threefold classification
of critical = stationary points = sta pts
of a real function of one variable:
maximum point = max pt
inflection point = flex pt
minimum point = min pt

• there are three infinities
associated with the real number line:
plus infinity, minus infinity, projective infinity;
plus infinity and minus infinity come from
the two-point compactification of the real line;
projective infinity comes from
the one-point compactification of the real line

• there are incomplete elliptic integrals of
the first, second, and third kinds

• there are Bessel functions of
the first, second, and third kinds

• the three boundary value problems
(first = Dirichlet, second = Neumann, third)
for harmonic functions

• the general third degree polynomial equation
in one variable over the complex field
is solvable by radicals using the coefficients

• the three types of isolated singularities
of a complex analytic function:
removable singularity, pole, essential singularity

• Hadamard's three-circle theorem in complex analysis

• the three fundamental forms (first, second, third)
of a hypersurface in euclidean n-space

• in differential geometry and tensor analysis
appear the Christoffel three-index symbols
of the first and second kind

• in the theory of obstructions
there are three (first, second, third)
classification theorems,
extension theorems,
homotopy theorems,
and there are three (primary, secondary, tertiary)
obstructions

• the three defining properties of an equivalence relation:
reflexive, symmetric, transitive
(mnemonic: rst,
three consecutive letters of the alphabet)

• the three defining properties of a partial order:
reflexive, antisymmetric, transitive

• the trichotomy law for
linearly/totally ordered sets = losets = tosets:
exactly one of these statements holds:
$x < y$ or $x = y$ or $x > y$

• the three basic binary operations in a ring:
addition, subtraction, multiplication

• the three isomorphism theorems (first, second, third)
for topological groups

• the threefold classification of cardinality:
finite
countably infinite = denumerable
uncountably infinite = uncountable

• the law of the excluded middle:
tertium non datur (Latin)
= lit: the third (case) is not given
= a proposition is true or false
= p or not p
= $p \vee \neg p$

• the threefold
validity classification
of statements as:
valid, nonsatisfiable, contingent

• the threefold
provability classification
of statements as:
provable, disprovable, undecidable

• three notable theorems of Gödel are
(1) Gödel's Completeness Theorem.
The lower predicate calculus (= first-order logic) is complete.
(2) Gödel's First Incompleteness/Undecidability Theorem.
Any consistent formal system rich enuf to contain arithmetic
contains undecidable propositions.
(3) Gödel's Second Incompleteness/Undecidability Theorem.
In any consistent formal system rich enuf to contain arithmetic
the proposition that the system is consistent
is undecidable.

• there is a three-valued logic among n-valued logics
(which has been used in an attempt to explain
quantum mechanics)

• the three-body problem of celestial mechanics;
unsolved at present (2000)

• the restricted three-body problem of celestial mechanics;
much known about this special case

• the 3 X problem where
X = container, gallon, glass, jar, jug, etc

• the power with base x and exponent three
= the power with base x and exponent 3
= the third power of x
= the 3rd power of x
= the 3rd pow of x
= x to the third power
= x to the 3rd power
= x to the 3rd pow
= x to the third
= x to the 3rd
= x to the power three
= x to the power 3
= x to the pow 3
= the cube of x
= x cubed
= x cube
= x^3

• the power with base x and exponent one-third
= the power with base x and exponent 1/3
= the one-third power of x
= the 1/3 power of x
= the 1/3 pow of x
= x to the one-third power
= x to the 1/3 power
= x to the 1/3 pow
= x to the one-third
= x to the 1/3
= x to the power one-third
= x to the power 1/3
= x to the pow 1/3
= the cube root of x
= $\sqrt[3]{x}$

• an X of degree three
= an X of degree 3
= an X of deg 3
= a third degree X
= a 3rd degree X
= a 3rd deg X
= a cubic X
where
X = equation, form, polynomial, differential equation, etc

• an X of dimension three
= an X of dimension 3
= an X of dim 3
= a three-dimensional X
= a 3-dimensional X
= a 3-dim X
= a 3-D X
= a 3-X (sometimes)
where
X = topological space, vector space, manifold, geometric object, geometry, etc

• an X of index three
= an X of index 3
where
X = almost anything, notationally an adscript

• an X of order three
= an X of order 3
= an X of ord 3
= a third order X
= a 3rd order X
= a 3rd ord X
= a three-by-three X
= a 3 by 3 X
= a 3×3 X
where
X = matrix, determinant, magic square, etc

• an X of order three
= an X of order 3
= an X of ord 3
= a third order X
= a 3rd order X
= a 3rd ord X
where
X = form, derivative, differential equation, etc

• an X of rank three
= an X of rank 3
= an X of rnk 3
where
X = form, determinant, group, Lie algebra, Lie group, map, matrix, tensor, etc

• an X of rank three
= an X of rank 3
= an X of rnk 3
= a third rank(ed) X
= a 3rd rank(ed) X
= a 3rd rnk X
where
X = datum, process, statistical result, etc

• the three parts of
the Tree of Mathematics:
roots = axioms
trunk = proofs
leaves = theorems

• the three big ems of statistics:
mean, median, mode

• the three material constituents of an atom:
protons, neutrons, electrons

• the three great frontiers of science:
the very big
the very small
the very complex

• it is likely that a coherent triplet
is easier to understand and remember than
three disparate singlets

□ a triangle has many triplets

altitudes
altitude-feet
angles = interior/internal angles
angle bisectors = interior/internal angle bisectors
exterior/external angles
exterior/external angle bisectors
excircles
excenters
exradii
medians
symmedians
pedal points of a point
sides
side-lenghts
side-midpoints
side-perpendicular-bisectors
vertices

etc

□ a terrific triplet of theorems on triangles

• the nine-point circle theorem:
for any triangle
the following three triples of notable points
all lie on a circle
called the nine-point circle:
the midpoints of the three sides,
the feet of the three altitudes,
the midpoints of the three vertex-to-orthocenter segments

• Feuerbach's theorem (1822):
for any triangle
the nine-point circle
is tangent to
the inscribed circle internally
&
the three exscribed circles externally

• bioline
Karl Wilhelm Feuerbach
1800-1834
German
geometer

• Morley's theorem (ca 1899):
for any triangle
the three pairs of adjacent internal angle trisectors
intersect in the vertices of
an equilateral triangle

• bioline
Frank Morley
1860-1937
English-American
algebraist, geometer;
the American writer
Christopher Morley (1890-1957)
was his son

☐ three identities for the three angles of a triangle

- $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

- $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

- $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

□ the three-point surveying problem

given three collinear points A, B, C
with known distances between them
and
given a point S with known angles ASB and BSC,
to find the distance SB;
this is also the problem of finding the distance
from a ship S at sea to the reference point B on shore;
or more generally
to find the point from which
pairs of three given points
are seen under given angles;
or equivalently
to find the point from which
the sides of a given triangle
are seen under given angles

□ the two triangles of three dots

Δ the up triangle of three dots $\therefore$
means
therefore, hence

Δ the down triangle of three dots $\because$
means
since, because

Δ $\therefore$ & $\because$
are kinds of converses of each other
because

- $\therefore$ is analogous / related to $\Rightarrow$

- $\because$ is analogous / related to $\Leftarrow$

- $p \therefore q \Leftrightarrow q \because p$

☐ the three solids of Cézanne:
the cylinder, the sphere, the cone

in 1904 the French Postimpressionist painter
Paul Cézanne (1839-1906) said

'Nature must be treated in terms of
the cylinder, the sphere, and the cone.'

with his own painting and this statement
he became the forerunner
of Cubism (ca1907-ca1915)
which was the most influential
of all modern art movements
and out of which grew
most of the earlier forms of abstract art

□ the three Frenet-Serret formulas
are
the central formulas in the theory of space curves

$$\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n} = \mathbf{d} \times \mathbf{t}$$

$$\frac{d\mathbf{n}}{ds} = -\kappa\mathbf{t} + \tau\mathbf{b} = \mathbf{d} \times \mathbf{n}$$

$$\frac{d\mathbf{b}}{ds} = -\tau\mathbf{n} = \mathbf{d} \times \mathbf{b}$$

wh

s = arclength

κ = curvature

τ = torsion

$\mathbf{t}$ = unit tangent vector

$\mathbf{n}$ = unit principal normal vector

$\mathbf{b}$ = unit binormal vector

$\mathbf{d}$ = the Darboux rotation vector = $\tau\mathbf{t} + \kappa\mathbf{b}$

GG42-54

bioline
Jean Frédéric Frenet
1816-1900
French
differential geometer, astronomer

bioline
Joseph Alfred Serret
1819-1885
French
analyst, differential geometer, number theorist, astronomer

□ the three greatest mathematicians of all time
in chronological order

• Archimedes
ca 287-212 BCE
Greek

• Newton
1642-1727
English

• Gauss
1777-1855
German

their full names

• Archimedes of Syracuse

• Isaac Newton

• Carl Friedrich Gauss

GG42-56

□ the three greatest geometers of antiquity were all Greek

• Euclid of Alexandria
fl ca 300 BCE

• Archimedes of Syracuse
ca 287 - 212 BCE

• Apollonius of Perga
ca 255 - ca 170 BCE

this listing is in chronological order

☐ the three most prolific mathematicians of all time
in decreasing quantitative order

• Euler
1707-1783
Swiss, lived in Germany & Russia

• Cauchy
1789-1857
French

• Cayley
1821-1895
English

their full names

• Leonhard Euler

• Augustin-Louis Cauchy

• Arthur Cayley

GG42-58

□ the three ancient A's
were all Greek

• Aristotle of Stagira
384-322 BCE
philosopher & scientist;
one of the most important & influential figures
in Western civilization

• Archimedes of Syracuse
ca 287-212 BCE
mathematician & physicist;
one of the three greatest mathematicians
of all time

• Apollonius of Perga
fl 250-220 BCE
mathematician;
called 'The Great Geometer'

□ the three L's
in the history of mathematics

Lagrange 1736 - 1813

Laplace 1749 - 1827

Legendre 1752 - 1833

were the principal French analysts
at the time of

the French Revolution 1789 - 1795

&

the Napoleonic Era 1796 - 1815□□□

□ the simplest 3 by 3 magic square

• the Lo-shu
is
the following 3×3 magic square

4	9	2
3	5	7
8	1	6

• the Lo-shu is the simplest of all magic squares;
its entries are the first nine positive integers
and it is essentially unique (except for reflections)

• the Lo-shu
has
magic constant
= 15
= the sum of each of the three rows
= the sum of each of the three columns
= the sum of each of the two diagonals

• there are many patterns to be found in the Lo-shu;
here are some of them

• the least number 1 is the center entry of the bottom row;
the greatest number 9 is the center entry of the top row

• the middle entry 5
= the middle number between 1 & 9
= the arithmetic mean of 1 & 9

• the isosceles triangle 1-2-3
(base at 1-3 & opp vertex at 2)
= the arrowhead 1-2-3 (tip at 2)
is formed by
the first triple of numbers between 1 & 9
and points to the north-east

• the isosceles triangle 7-8-9
(base at 7-9 & opp vertex at 8)
= the arrowhead 7-8-9 (tip at 8)
is formed by
the last triple of numbers between 1 & 9
and points to the south-west

• the principal diagonal 4-5-6
= the middle triple of numbers between 1 & 9

• the odd number entries form a cross in the middle

• the even number entries are at the four corners

• z-shaped patterns are formed
by the consecutive odd number entries 1-3-5-7-9
&
by the consecutive even number entries 2-4-6-8

• the Lo-shu
is the oldest known example of a magic square;
the Lo-shu
may be called
the Chinese turtle magic square
because
Chinese mythology alleges that the Lo-shu was first seen
by the great Emperor Yu around 2200 BCE
as a decoration on the back of a divine turtle
appearing on the bank of the Yellow River (= Lo in Chinese)
when he was embarking onto the river

• the Lo-shu & the associated legend
are found in the I-Ching
= pr ee-king
which is an ancient Chinese book on divination
& which contains systematic permutations;
the book was probably written about 1130 BCE

□ the three big C's of general topology
are the three topological properties
which are listed below in the form
• adjective
  noun

• continuous
  continuity

• compact
  compactness

• connected
  connectedness

where

• continuous
= preserves nearness
which refers to a function
from a topological space to a topological space

• compact
= the topological generalization of finite
which refers to a topological space

• connected
= the precise topological description of being in one piece
which refers to a topological space GG42-65

□ three classes of sets that are not sets

• the class of all sets
= the class of sets
= the set class
= Set

• the class of all ordinals
= the class of ordinals
= the ordinal class
= Ord

• the class of all cardinals
= the class of cardinals
= the cardinal class
= Crd

• Set ... Ord ... Crd, none of which are sets

• instead of saying
set_0 , set_1 , set_2 , $\cdots$,
say
set, class, collection, $\cdots$;
everything is still a 'set';
the distinction among
'set', 'class', 'collection', $\cdots$
is just a notational/terminological device
for the sake of clarity & simplicity
note: ordinal = ordinal number & cardinal = cardinal number
GG42-66

☐ a three category classification
in which
one class appears to be
somewhat special
&
serving to separate the other two classes
often receives the designations
• of elliptic type
• of parabolic type
• of hyperbolic type
(the three classes usually have nothing to do
with conic sections themselves)
because of
the prototype classification
of conic sections
viz
• ellipses
have eccentricity $e < 1$
• parabolas
have eccentricity $e = 1$
• hyperbolas
have eccentricity $e > 1$

examples of this kind of classification include
• geometries
• points on a surface
• surfaces of revolution of constant curvature
• simply connected Riemann surfaces
• 2nd order PDE's

☐ the three little words
constant/variable/parameter
¿ what's the difference ?

for a given mathematical discourse:

• a constant
= a symbol with a single value assigned

• a variable
= a symbol with a set of values assigned,
this set being called
the range of the variable

• a parameter
= a symbol which is
sometimes considered to be a constant
&
sometimes considered to be a variable

• the value of a constant
= the unique object
that it is assumed the constant stands for

• a value of a variable
= an element of the range of the variable,
the range being the set of all objects
that it is assumed the variable may stand for

¡ vive la différence ! GG42-68

☐ the three -jection words for functions

- injection/injective = one-to-one
- surjection/surjective = onto
- bijection/bijective = one-to-one onto

other -jection words in the English language include

dejection
ejection
interjection
introjection
objection
projection
rejection
subjection

☐ names of unit vectors

2-dim unit vectors

- $\mathbf{i} = (1, 0) =$ Little Isaac
- $\mathbf{j} = (0, 1) =$ Little Jacob

3-dim unit vectors

- $\mathbf{i} = (1, 0, 0) =$ Isaac
- $\mathbf{j} = (0, 1, 0) =$ Jacob
- $\mathbf{k} = (0, 0, 1) =$ Kilroy

n-dim unit vectors wh $n \in$ pos int

- $\mathbf{e}_1 = (1, 0, 0, \cdots, 0, 0) =$ First Elf
- $\mathbf{e}_2 = (0, 1, 0, \cdots, 0, 0) =$ Second Elf

$\vdots$

- $\mathbf{e}_n = (0, 0, 0, \cdots, 0, 1) =$ nth Elf

the letter e comes from the German word
die Einheit = unit/unity

☐ the three old -chrones

- brachistochrone
- isochrone
- tautochrone

brief descriptions

- brachistochrone

= curve of shortest descent
= cycloid

- isochrone

= curve of equal descent
= cycloid

- tautochrone

can mean
isochrone
or something else

- -chrone (= time)

≠ -chrome (= color)

GG42-71

a lesson in Greek & in etymology

Greek	part of speech	meaning
• βραχυς	adjective	short
• ισος	adjective	equal
• το αυτο	pronoun	the same
• χρονος	noun	time
• χρωμα	noun	color

☐ the three most overused words in mathematics

generally mathematicians
are not known to be
creative and ingenious
when it comes to thinking up
an apt previously unused word
to name a just-defined object/property/method/whatever;
it is often the case that they select
some word that has been used
many times in other contexts;
it is likely that
the three most overused words in mathematics
are the three given below;
each word is overused in the sense that
each has many different meanings
dependent on the context

- conjugate

- normal

- regular

it is frequently the case that
a mathematician will choose a nice word
suggesting this is the way things should be
for the situation they can handle or are interested in
and choose the opposite sort of word
for the opposite situation;
hence
eg

the nice words:
normal
regular
simple
smooth
stable
standard
tame

the no-so-nice words:
chaotic
pathological
rough
singular
sporadic
turbulent
unstable
wild

GG42-74

□ three grams/graphs/signs

• ideogram
= ideograph
= idea-sign
= a sign by which an idea is written

• logogram
= logograph
= word-sign
= a sign by which a word is written

• syllogram
= syllograph
= syllable-sign
= a sign by which a syllable is written

☐ three -ati words

- digerati = computerly people
- illuminati = enlightened people
- literati = scholarly people

also

- intelligentsia = intellectual people

GG42-76

☐ three good foreign words
adopted into English;
each is extraordinaire (French) = extraordinary

• aficionado/a (Spanish)
= pr ah-FEES-ee-oh-NAH-doh/dah
= a male/female person
who appreciates/knows/likes
an avidly pursued activity/interest

• cognoscente/i (Italian)
= pr kahn-yuh-SHEN-tuh/tee
= person/persons who is/are
especially knowledgeable in a subject

• connoisseur (French)
= pr KAH-nuh-SURR (English)
= an expert in a subject who enjoys it
with a discriminating taste
& an appreciation of subtleties

□ three types of languages

• SVO language
= a language that has the basic
subject-verb-object order
in a sentence
eg English, Spanish, Chinese

• SOV language
= a language that has the basic
subject-object-verb order
in a sentence
eg Turkish, Japanese, Tamil

• VSO language
= a language that has the basic
verb-subject-object order
in a sentence
eg Welsh, classical Arabic, Tagalog

☐ three good words for each language X

- Xphile

= one who loves the X language and X things

- Xphobe

= one who hates the X language and X things

- Xphone

= one who speaks the X language

where possibly X = math

□ Kepler's three laws of planetary motion

• Law 1. The orbit of each planet is an ellipse
with the sun at one of its foci.

• Law 2. A line connecting any planet with the sun
sweeps over equal areas in equal time
during orbital motion.
More briefly,
the areal speed of each planet is constant.

• Law 3. The square of the period of revolution of any planet
is proportional to
the cube of the major axis of the planet's elliptical orbit,
the constant of proportionality
being the same for all planets.

bioline
Johannes Kepler
1571-1630
German
astronomer, mathematician, philosopher

□ Newton's three laws of motion

• Law 1. Every object remains at rest
or moves with constant speed in a straight line
unless acted upon by an external force.

• Law 2. For an object in motion
force equals mass times acceleration.
More fully,
the vector force
equals
the time rate of change of the vector momentum.

• Law 3. If one object exerts a force on a second object,
then the second object exerts a force on the first object
that is equal in magnitude and opposite in direction.
More briefly,
to every action there is an equal and opposite reaction.

bioline
Isaac Newton
1642-1727
English
mathematician, physicist;
one of the three greatest mathematicians of all time,
the other two being Archimedes and Gauss

☐ the three laws of thermodynamics
which are
restated in
the language of the theory of games
&
applied to
the game of life

Law 1. You can't win.

Law 2. You can't break even.

Law 3. You can't get out of playing the game.

GG42-82

• thermodynamics is the branch of physics
that studies heat

• first law of thermodynamics
= the law of conservation of energy
= energy can neither be created nor destroyed
but only converted from one form into another form

• second law of thermondynamcs
= entropy increases
= heat can flow only from a warmer body to a cooler body

• third law of thermodynamics
= every system has a finite positive entropy
at a temperature above absolute zero
but its entropy may become zero
at a temperature of absolute zero

• entropy
= in a closed system
the measure of the unavailability
of the thermal energy to do mechanical work
= measure of the degree of disorder of any system

☐ the three ways of doing modern science

(1) theory

(2) experimentation & observation

(3) computer simulation/graphics

in the past
mathematics
was virtually confined to (1)
but now
mathematics
can substantially participate in (3)

□ the three-fold spherical shell structure of the earth

Δ the crust of cool hard rock

Δ the mantle of hot plastic rock

Δ the core of hot iron

- the outer core of hot liquid iron

- the inner core of hot solid iron

in brief
Δ crust
Δ mantle
Δ core
- outer core
- inner core

□ three-tier tidbits to teachers

Δ the three I's of instruction

the task of the teacher
is to provide
the three I's of instruction
viz
• information
• insight
• inspiration

a teacher
= a purveyor of the three I's
= a person with three eyes
= a three-eyed person

Δ the three I's of discovery/invention

the three steps to discovery/invention
as described by Helmholtz
are
the three I's of discovery/invention
viz
• immersion
• incubation
• illumination

Δ three educational desiderata:

- literacy = to be literate
- numeracy = to be numerate
- computeracy = to be computerate

Δ the three schoolhouse R's:

- Reading
- wRiting
- aRithmetic

or more alliteratively

- readin'
- 'ritin'
- 'rithmetic

which are the three principal subjects traditionally taught in elementary school

Δ advice to teachers:

you can teach technique
but
you can't teach talent

since
technique is environmental
&
talent is genetic

Δ bioline
Hermann Ludwig Ferdinand von Helmholtz
1821 - 1894
German
physicist, physiologist, applied mathematician

☐ the three slices of the math pie

• mathematics may be divided up initially
into three principal branches
viz
algebra
analysis
geometry/topology

• if mathematics is thought of
as represented by a circular disk
and
if the three principal branches of mathematics
are thought of as three equal circular sectors
then
the three bounding radii and the center
are wide and full of content
ie
much mathematics is a mixture
of some two of these branches
and
much mathematics is a mixture
of all three of these branches

• algebra may be defined as
the study of finitary operations and relations

• analysis may be defined as
the study of the limit properties
of numbers and functions of numbers

• topology may be defined as
the study of the general notion of limit

• to a mathematician these three brief descriptions
present a clear operationally feasible method
for classifying any given piece of mathematics as
algebra or analysis or topology
or some mixture of these

• the situation with respect to geometry
is not so simple;
in fact no such simple prescription of what geometry is
has ever been recognized;
for a detailed discussion of this fact
and why geometry and topology together
should be considered as constituting
a single principal branch of mathematics
rather than two or more principal branches,
see packet #17 What Is Geometry?

☐ words meaning three, abstractly or concretely,
or pertaining thereto
and of mathematical use

clover
cubature
cube
cubic
cubical
cubiform
cuboctahedron
cuboid
delta
deltahedron
delta-shaped
deltoid
deltoidal
fan-shaped
n-thirds wh n = a positive integer
one-third
rhombicuboctahedron
ternary
ternion

third
thirdly
three
three-angled
three-cornered
three-dimensional
threefold
three-forked
three-nths wh n = a positive integer
three-part
three-phase
three-ply
three-pronged
three-quarter
threescore
three-sided
three-valued
three-X wh X = a noun or an adjective

trefoil
triacontahedron
triad
triadic
triality
triangle
triangulable
triangular
triangulate
trianglulation
triaxial
triclinic
trichotomize
trichotomous
trichotomy
tricornered
trident
trifold
trifolium
trifurcate
trifurcation
trigon
trigonal
trigonometric
trigonometry
trigram
trigraph
trihedral
trihedron

trilateral
trilemma
trilinear
triliteral
trilogic
trilogical
trinomial
trinormal
trionym
tripartite
tripartition
triple
triplet
triplex
triplicate
triplication
triplicity
triply
tripod
trirectangular
trisect
trisected
trisecting
trisection
trisector
trisectrix
trisoctahedron
trisyllable
trivium
two-thirds

also in context:

tertium non datur (Latin)
= lit: third not given
= law of the excluded middle
cubical parabola
semicubical parabola
three-body problem
three R's
3-D

□ What I tell you three times is true.

the above line is a quotation from
Lewis Carroll's poem
Hunting of the Snark

bioline
Lewis Carroll
was the pseudonym of
Charles Lutwidge Dodgson
1832-1898
English
writer of children's stories,
Oxford University mathematics don,
photographer

Charles is an English and French name of Germanic origin;
Charles is used an an anglicized form
of the Irish Gaelic name Cearbhall;
Carroll is also an anglicized form of Cearbhall;
Lewis is a more anglicized form
of the English name Lutwidge of Germanic origin

☐ three mathematical cornerstones

• the cornerstone of geometry
= Euclid's book
'Elements' ca 300 BCE in Greek

• the cornerstone of algebra
= al-Khwarizmi's book
'Algebra' ca 800 CE in Arabic

• the cornerstone of analysis
= Euler's book
'Introduction' 1748 CE in Latin

□ the three principal principles/procedures of mathematics

- make definitions
- prove theorems
- construct examples

more briefly

- define
- prove
- exemplify□□

☐ human duality in three languages

English	Latin	Greek
body	corpus	σωμα (soma)
&	&	&
mind / soul	mens / animus	νους / ψυκη (nous / psyche)

□ the occurrences in nonmath contexts
of ideas and words alluding to three
are beyond easy counting and listing

here is a more or less random sampling
containing something over 50 items
with brief explanations/identifications

• tercentenary
= tercentennial
= tricentennial
= adjective or noun
for a 300th anniversary or its celebration

• tercet
= triplet
= a stanza of poetry consisting of three equal-length lines
often rhyming together

• ternate
= describing a compound leaf as
divided into three equal parts

• tertian
= recurring every other day
= every third day inclusive

• tertiary
= third in degree/importance/order/place

• tertium quid (Latin)
= lit: third something
= something that serves as a compromise
between two oppositie things;
sometimes in philosophy
something neither mind nor matter

GG42-101

• the traditional three ages of prehistory
= Stone Age, Bronze Age, Iron Age;
a more modern classification runs

Old Stone Age	= Paleolithic
Middle Stone Age	= Mesolithic
New Stone Age	= Neolithic
Copper Age	= Chalcolithic
Bronze Age	
Iron Age	

• the three degrees of comparison
of adjectives and adverbs:
positive, comparative, superlative

• three-dog night
= a very cold night
as measured by the number of dogs
one has to have in the bed
in order to keep warm

• the three Fates of Greek mythology
= three old women,
daughters of Nyx = Night,
who
spin (Clotho), measure (Lachesis), cut (Atropos)
the thread of life

• the three Furies of Greek mythology
= Alecto, Megaera, Tisiphone

• the three Graces of Greek mythology
= Aglaia (Brilliance), Euphrosyne (Joy), Thalia (Bloom)

• the three Horae (Latin for 'hours') of Greek mythology
= Dike (Justice), Eunomia (Order), Irene (Peace)

• the three Sirens of Greek mythology
= Leucosia, Ligea, Parthenope

• the three goddesses of Greek mythology
from whom Paris had to choose the most beautiful
= Athena, Aphrodite, Hera

• the three kingdoms of nature
= animal, vegetable, mineral

• The Three Musketeers
= Athos, Porthos, Aramis
from the historical novel (1844)
of the same name
Les Trois Mousquetaires (French)
by the French writer Alexandre Dumas père
(d'Artagnan was the fourth musketeer)

• The Threepenny Opera
= Die Dreigroschenoper (German)
= ballad-opera (1928, tr 1933)
by the German writer Bertolt Brecht
with music by the German-American composer Kurt Weill
(based on John Gay's The Beggar's Opera)

• the three primary colors
=
red, green, blue (additive primaries, for light)
or
cyan, magenta, yellow (subtractive primaries,
for pigments/photography/printing)

• The Three Princes of Serendip
= Persian fairy tale which suggested (1754)
the word serendipity
to the English writer Horace Walpole
(Serendip = old Arabic name of Sri Lanka)

• three-ring circus
= a circus with three rings in which
there are simultaneous performances
and thus by extension
an extravagant display

• three score years and ten
= biblical (KJV) phrase for the expected length of human life

• the three ships of Christopher Columbus
on his first voyage to America in 1492
= Niña, Pinta, Santa Maria

• the three Brontë sisters
= Anne (1820-1849)
Charlotte (1816-1855)
Emily (1818-1848)
who hold a remarkable place
in English literary history

• The Three Sisters
= a play (1901) by the Russian writer Anton Chekhov

• the Three Sisters
= three adjacent mountain peaks in western Oregon,
all over 10,000 feet high

• the three Weird Sisters
= the three witches
in Shakespeare's play Macbeth (ca 1606)

• thrice
= three times

• trefoil
= three-leaved plant or ornament

• trialogue
= colloquy among three people

• Triangulum
= a small constellation in the northern hemisphere
(Latin for triangle)

• Triangulum Australe
= a small constellation in the southern hemisphere
(Latin for southern triangle)

• trianthous
= having three flowers

• triarchy
= government by three persons

• triathlon
= athletic contest with three events

• tricolor
= a national flag of three broad stripes of different colors

• tricorne
= a three-cornered hat
with the brim turned up
on all three sides

• trident
= in classical mythology
the three-pronged spear that
the Greek Poseidon = the Roman Neptune,
god of the sea,
carried as the symbol of his authority

• trifecta
= a bet on a horse race
in which the bettor must predict
the first three finishers in exact order

• the Trifid Nebula
= a bright nebula in Orion
(trifid is from the Latin word trifidus = split in three
& nebula is a Latin word meaning cloud/fog/mist)

• trifold
= threefold

• triform
= having three bodies/parts/shapes

• trilogy
= a sequence of three literary/musical
self-contained works relating to a common theme

• triphthong
= vowel sound with three elements in one syllable

• Triple Crown
= victory in all three horseracing events
Kentucky Derby, Preakness Stakes, Belmont Stakes
in the same season

• tripos
= a Cambridge University term meaning
the three honor classes into which
bachelor degree candidates
were grouped after the final examination
in various subjects including mathematics;
so-called because
the oral examiner in the School of Philosophy
used to sit on a tripus (Latin) = three-legged stool
and was called 'Mister Tripos'

• triptych
=
work of art in three connected pieces
or
three connected writing tablets used in ancient times
(from a Greek word meaning threefold)

• triskaidekaphobia
= morbid fear of the number 13
(from Greek meaning three-and-ten-fear)

• triskelion
= symbol in the form of three bent lines/limbs
radiating from a common center
(from a Greek word meaning three-legged)

• Hermes Trismegistus (Greek word)
= Hermes thrice greatest
(meaning three times greater than
the Greek god Hermes)
= an Egyptian priest
or Thoth, the Egyptian god of wisdom,
who was fabled to have dictated
forty-two books dealing with
the life and thought of ancient Egypt

Hermes
in ancient Greek mythology
was
the son of Zeus & Maia;
he was the god of
commerce, eloquence, invention, roads, travel, and theft;
he was the herald/messenger of the gods, and
the bearer of the symbolic staff of the herald
which is called the 'caduceus'
(from the Greek word καρυξ meaning 'herald')
and which consists of a staff
with two entwined serpents
and two wings on top;
the caduceus is now the modern physician's ensignia;
Hermes was identified with Mercury by the Romans

- tritone

= musical interval consisting of three whole tones

- triumvirate

= a group of three people who act jointly
for some special, usually governmental, purpose;
in ancient Roman history
The First Triumvirate
= Julius Caesar, Pompey, Crassus
in 60 BCE
&
The Second Triumvirate
= Octavius (later Caesar Augustus), Antony, Lepidus
in 43 BCE

• trivalent
= having a chemical valence of three

• troika
= a Russian word meaning
a team of three horses abreast
with or without a drawn vehicle
and by extension a set of three

• the eternal triangle
= amorous involvement of three people

• Twenty-three skiddoo!
= slang exclamation of variable meaning
highly popular ca 1900 -1910

• the number three occurs many times in
nursery rhymes/songs/stories
eg
Goldilocks and the three bears
I saw three ships come sailing by
the three little kittens
the three little pigs
three men in a tub
etc

□ the three binomial formulas/theorems

• the binomial formula/theorem
for ordinary powers

$$(a+b)^{n} = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}\, b^{r}$$

• the binomial formula/theorem
for rising factorial powers

$$(a+b)^{\overline{n}} = \sum_{r=0}^{n} \binom{n}{r} a^{\overline{n-r}}\, b^{\overline{r}}$$

• the binomial formula/theorem
for falling factorial powers

$$(a+b)^{\underline{n}} = \sum_{r=0}^{n} \binom{n}{r} a^{\underline{n-r}}\, b^{\underline{r}}$$

wh
a, b $\in$ complex nr
&
n $\in$ nonneg int

GG42-113